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Cordial Labelling Of 3-Regular Bipartite Graphs

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Abstract

In the labelling of graphs one of the types of labelling is cordial labelling. In this we label the vertices 0 or 1 and then every edge gets a label 0 or 1 according to the labels of its end vertices. If the end vertices of an edge have same labelling then the edge gets the labelling 0 and if the end vertices have different labelling then the edge gets the labelling 1. After labelling the given graph in this way, if | number of vertices labelled '0' – number of vertices labelled '1' | = 0 Or 1 and also| number of edges labelled '0' – number of edges labelled '1' | = 0 Or 1 then this kind of labelling is called a cordial labelling and the graph is said to be cordial. Here we are going to prove a 3-regular bipartite graph is cordial.

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I. INTRODUCTION

We will deal with simple graphs only.

- Regular graph: A graph G is said to be a regular graph if degree of each vertex is same. It is called k-regular if degree of each vertex is k.
- Bipartite graph: Let G be a graph. If the vertices of G are divided into two subsets A and B such that there is no edge 'ab' with $a,b \in A$ or $a,b \in B$ then G is said to be bipartite that is, every edge of G joins a vertex in A to a vertex in B.
  The sets A and B are called partite sets of G.
- Complete graph: A graph in which every vertex is adjacent to every other vertex is a complete graph. For a complete graph on n vertices degree of each vertex is n-1.
- Complete bipartite graph: A bipartite graph G with bipartition (A,B) is said to be complete bipartite if every vertex in A is adjacent to every vertex in B. It is easy to see that if G is k-regular bipartite graph then | A | = | B |
- Cycle: A closed path is called a cycle.
- Labelling of a graph:
  Vertex labelling: It is a mapping from set of vertices to set of natural numbers.
  Edge labelling :It is a mapping from set of edges to set of natural numbers .
- Cordial labelling: For a given graph G label the vertices of G by '0' or '1'. and every edge $'ab'$ of G will be labelled as '0' if the labelling of the vertices 'a' and 'b' are same and will be labelled as '1' if the labelling of the vertices 'a' and 'b' are different. Then this labelling is called a "cordial labelling" or the graph G is called a "cordial graph" iff,

$$\left| No.\ of\ \ vertices\ labeled\ \ '0' - No.\ of\ \ vertives\ labeled\ \ '1' \right|$$
$$\leq 1$$
$$And$$
$$\left| No.of\ \ edges\ labeled\ \ '0' - No.\ of\ \ edges\ labeled\ \ '1' \right|$$
$$\leq 1$$

II. A 3-REGULAR BIPARTITE GRAPH WITH PARTITE SETS A AND B IS CORDIAL FOR | A | = | B | = 4M

Let 'G' be a 3-regular bipartite graph with partite sets 'A' and 'B' with | A | = | B | = 4n
Let 2m vertices of A are labelled '1'and 2m are labelled as '0'
Also, 2m vertices of B are labelled '1'and 2m are labelled as '0'

NOTATION :

a_{10} ---- Number of edges 'ab' where $a \in A, b \in B$
with 'a' is labeled as 1 and 'b' is labeled as '0'

a_{11} ---- Number of edges 'ab' where $a \in A, b \in B$
with 'a' is labeled as 1 and 'b' is labeled as '1'

a_{01} ---- Number of edges 'ab' where $a \in A, b \in B$
with 'a' is labeled as 0 and 'b' is labeled as '1'

a_{00} ---- Number of edges 'ab' where $a \in A, b \in B$
with 'a' is labeled as 0 and 'b' is labeled as '0'

Then we have;

$$\left.\begin{aligned} a_{11} + a_{10} &= 6m \text{ ------(1)} \\ a_{01} + a_{00} &= 6m \text{------(2)} \\ a_{01} + a_{11} &= 6m \text{------(3)} \\ a_{00} + a_{10} &= 6m \text{------(4)} \end{aligned}\right\} \text{----------------*}$$

Define $\beta =$ difference between no. Of vertices labelled '0' and no. Of vertices labelled '1'

Claim: β = 0, 4, 8, 12,…12m

First we will prove,

$$\beta = 0 \quad \text{iff} \quad \begin{matrix} a_{11} = 3m & a_{01} = 3m \\ a_{10} = 3m & a_{00} = 3m \end{matrix}$$

If $a_{11} = 3m$, $a_{01} = 3m$, $a_{10} = 3m$, $a_{00} = 3m$

Then clearly, $\beta = \left|(a_{11} + a_{00}) - (a_{01} + a_{10})\right| = 0$

Conversely,

Let $\beta = 0$

$\Rightarrow \left|(a_{11} + a_{00}) - (a_{01} + a_{10})\right| = 0$

$\Rightarrow a_{11} + a_{00} = a_{10} + a_{01}$ --------------------(A)

But from equations *

(1) & (4) $\Rightarrow a_{11} = 6m - a_{10}$ & $a_{00} = 6m - a_{10}$

$\Rightarrow a_{11} = a_{00}$ Also,

(1) & (2) $\Rightarrow a_{10} = 6m - a_{11}$ & $a_{01} = 6m - a_{00}$

$\Rightarrow a_{10} = a_{01}$ -------as $a_{11} = a_{00}$ proved earlier.

$\therefore$ (A) gives, $a_{11} = a_{10}$

$\Rightarrow a_{11} = a_{00} = a_{10} = a_{01}$

but, $a_{11} + a_{00} + a_{10} + a_{01} = 3m$

$\Rightarrow a_{11} = a_{00} = a_{10} = a_{01} = 3m$

$$\therefore \beta = 0 \Rightarrow \begin{matrix} a_{11} = 3m & a_{10} = 3m \\ a_{01} = 3m & a_{00} = 3m \end{matrix}$$

Hence we have, $a_{11} = 3m \Leftrightarrow G \text{ is cordial}$

Now if

$a_{11} \neq 3m \ \ then\ a_{11} = 3m + i \quad for\ i = \pm 1, \pm 2, \pm 3, \ldots \pm 3m$

$First\ we\ will\ consider\ i = 1,2,3 \ldots as\ i = -1,-2,-3 \ldots$ can be proved similarly.

$\therefore consider\ a_{11} = 3m + i$

$\Rightarrow a_{01} = 3m + i \Rightarrow a_{00} = 3m + i \Rightarrow a_{10} = 3m + i$

$\Rightarrow \beta = 4i$

$\therefore$ The only values β can take are, 0,4,8,12,…4i,…

Now, if $\beta = 0$ then G is cordial

If $\beta = 4i\ then,$

$a_{11} = 3m + i \qquad a_{01} = 3m - i$

$a_{10} = 3m - i \qquad a_{00} = 3m + i$

We will interchange labelling of two vertices labelled 0 and 1 so that the count of β reduces by 4 and then by continuing the process repeatedly we get a labelling for which $\beta = 0$ and hence G becomes cordial.

NOTATION : $a-1, b-1, c-1$ means a vector in 'B' which is labelled as '1' has the three adjacencies in 'A' which are labelled as a,b,c.

Similarly, $a-0, b-0, c-0, \quad 0-a, 0-b, 0-c, \quad 1-a, 1-b, 1-c$ .Using this notation, consider any vector in B labelled as '1' then it has following four types of adjacencies,

(A) $1-1, 1-1, 0-1$ (B) $1-1, 0-1, 0-1$

(C) $1-1, 1-1, 1-1$ (D) $0-1, 0-1, 0-1$

And any vector in B labelled as '0' then it has following four types of adjacencies,

($A^{|}$)$1-0, 0-0, 0-0$ ($B^{|}$) $0-0, 0-0, 0-0$

($C^{|}$)$1-0, 1-0, 0-0$ ($D^{|}$) $1-0, 1-0, 1-0$

CLAIM: If we interchange the labelling of the vertices labelled 1 & 0 which have the adjacencies of the type A & $A^{|}$ OR B & $B^{|}$ OR C & $C^{|}$ respectively then count of β reduces by 4 and then by repeated application we get β = 0.

If the vertex labelled 1 with the adjacencies $1-1, 1-1, 0-1$ is changed to 0 it becomes $1-0, 1-0, 1-0$

If the vertex labeled 0 with the adjacencies $1-0, 0-0, 0-0$ is changed to 1 it becomes $1-1, 0-1, 0-1$

$\therefore a_{11} = 3m + i - 2 + 1 \qquad a_{01} = 3m - i - 1 + 2$

$a_{10} = 3m - i - 1 + 2 \qquad a_{00} = 3m + i - 2 + 1$

$i.e. a_{11} = 3m + i - 1 \qquad a_{01} = 3m - i + 1$

$a_{10} = 3m - i + 1 \qquad a_{00} = 3m + i - 1$

$$\therefore \beta = |(6m + 2i - 2) - (6m - 2i + 2)|$$
$$= 4i - 4$$

Similarly we get, By interchanging the vertices of the type B & $B^{|}$ OR C & $C^{|}$, β reduces by 4.

Now, Let

a = number of vertices in B labeled 1 which is of type A

b = number of vertices in B labeled '1' which is of type B

c = number of vertices in B labeled '1' which is of type C

d = number of vertices in B labeled '1' which is of type D

As we have,

$a_{11} = 3m + i \qquad a_{01} = 3m - i$

$a_{10} = 3m - i \qquad a_{00} = 3m + i$

$$\left.\begin{array}{ll} 2a+b+3c & =3m+i \\ a+2b\quad +3d & =3m-i \\ a+b+c+d & =2m \end{array}\right\}---------**$$

Case I: a and b both are zero.

The augmented matrix for the above non-homogenous system ** is,

$$[A\,|\,B]=\begin{bmatrix} 0 & 0 & 3 & 0 & | & 3m+i \\ 0 & 0 & 0 & 3 & | & 3m-i \\ 0 & 0 & 1 & 1 & | & 2m \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{1}{3}R_1}\begin{bmatrix} 0 & 0 & 3 & 0 & | & 3m \\ 0 & 0 & 0 & 3 & | & 3m \\ 0 & 0 & 0 & 1 & | & m-\frac{i}{3} \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{1}{3}R_2}\begin{bmatrix} 0 & 0 & 3 & 0 & | & 3m \\ 0 & 0 & 0 & 3 & | & 3m \\ 0 & 0 & 0 & 0 & | & -\frac{i}{3} \end{bmatrix}$$

$$\therefore\ rank\ of\ A=2\ and\ rank\ of\left[A|B\right]=3$$

$$\Rightarrow rank\ of\ A\neq\ rank\ of\left[A|B\right]$$

Hence the system does not have any solution.

Hence a and b both cannot be zero.

Case II: a and c both are zero.

The augmented matrix for the above non-homogenous system ** is,

$$[A\,|\,B]=\begin{bmatrix} 0 & 1 & 0 & 0 & | & 3m+i \\ 0 & 2 & 0 & 3 & | & 3m-i \\ 0 & 1 & 0 & 1 & | & 2m \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{1}{2}R_2}\begin{bmatrix} 0 & 1 & 0 & 0 & | & 3m+i \\ 0 & 2 & 0 & 3 & | & 3m-i \\ 0 & 0 & 0 & \frac{-1}{2} & | & \frac{m+i}{2} \end{bmatrix}$$

$$\xrightarrow{2R_3}\begin{bmatrix} 0 & 1 & 0 & 0 & | & 3m+i \\ 0 & 2 & 0 & 3 & | & 3m-i \\ 0 & 0 & 0 & -1 & | & m+i \end{bmatrix}$$

$$\xrightarrow{R_2-2R_1}\begin{bmatrix} 0 & 1 & 0 & 0 & | & 3m+i \\ 0 & 0 & 0 & 3 & | & -3m-3i \\ 0 & 0 & 0 & -1 & | & m+i \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{1}{3}R_2}\begin{bmatrix}0&1&0&0&|&3m+i\\0&0&0&3&|&-3m-3i\\0&0&0&0&|&0\end{bmatrix}$$

$$\because rank\ of\ A = rank\ of\left[A|B\right]=2$$

Hence solution exists.

This gives, $b=3m+i$

This is contradictions as $b\le 2m$

Again it does not have solution.

Hence a and c both cannot be zero.

Case III: b and c both are zero.

The augmented matrix for the above non-homogenous system ** is,

$$\left[A\,|\,B\right]=\begin{bmatrix}2&0&0&0&|&3m+i\\1&0&0&3&|&3m-i\\1&0&0&1&|&2m\end{bmatrix}$$

$$\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}1&0&0&3&|&3m-i\\2&0&0&0&|&3m+i\\1&0&0&1&|&2m\end{bmatrix}$$

$$\xrightarrow{R_2-2R_1}\quad\xrightarrow{R_3-R_1}$$

$$\begin{bmatrix}1&0&0&3&|&3m-i\\0&0&0&-6&|&-3m+3i\\0&0&0&-2&|&-m+i\end{bmatrix}\xrightarrow{R_3-\frac{1}{3}R_2}\begin{bmatrix}1&0&0&3&|&3m-i\\0&0&0&-6&|&-3m+3i\\0&0&0&-0&|&0\end{bmatrix}$$

$$\therefore rank\ of\ A = rank\ of\left[A|B\right]$$

Hence it has a solution.

$$a+3d=3m+i\ \ and$$

$$-6d=-3m+i$$

$$\Rightarrow d=\frac{3m-3i}{6}$$

$$\Rightarrow a+3\left(\frac{m-i}{2}\right)=3m-i$$

$$\Rightarrow 2a+3m-3i=6m-2i$$

$$\Rightarrow a=\frac{3m+i}{2}$$

Hence the solution is,

$$a=\frac{3m+i}{2}\ \ and\ \ d=\frac{m-i}{2}$$

Hence we have proved,

(A) a & b both can not be zero
a & c both can not be zero
b & c both of them may or may not be zero.

Similarly we can prove

(B) $a^{|}$ and $b^{|}$ both cannot be zero

$a^{|}$ and $c^{|}$ both cannot be zero

$b^{|}$ and $c^{|}$ both of them may or may not be zero.

From A and B :

When b & c both of them are not be zero and

$b^{|}$ and $c^{|}$ both of them are not be zero.

Then it gives,
Two of a,b,c are present and
Two of $a^{|}, b^{|}, c^{|}$ are present

$\Rightarrow$ one of the pairs $a \,\&\, a^{|}$, $b \,\&\, b^{|}$, $c \,\&\, c^{|}$ is present. Hence by exchanging their labelling β gets reduced by 4.

When b & c both of them are zero

$\Rightarrow$ all $1 \in$ B are of the a and d.

Now (i) if $\exists$ $0 \in \mathbf{B}^{|}$ of type $a^{|}$ then by exchanging $a \; and \; a^{|}$ β reduces by 4.

(ii) If there does not exist a vertex $0 \in B$ of type $a^{|}$

i.e. we have all $1 \in B$ of type a and d and

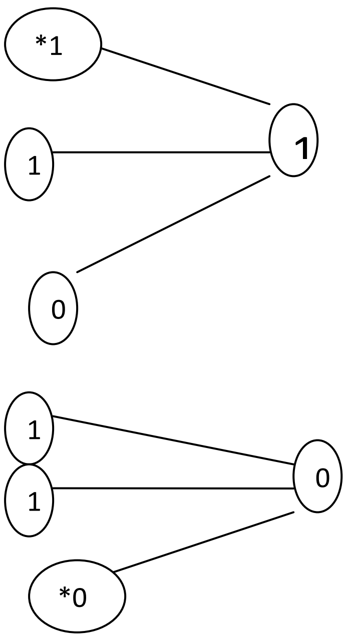
all $0 \in B$ of the type $b^{|}, c^{|}, d^{|}$

then, consider $1 \in B$ of type a i.e. 1-1,1-1,0-1 and

$0 \in B$ of type $C^{|}$ i.e. 1-0,1-0,0-0

(we always get $c^{|}$ as $a^{|}$ and $c^{|}$ both can not be zero)

That means we have vertices as shown,



Then by exchanging the labelling of 1 and 0 marked by * of elements of A this gives

$a_{11} = 3m + i - 1$ $\quad a_{01} = 3m - i + 1$

$a_{10} = 3m - i + 1$ $\quad a_{00} = 3m + i - 1$ This gives $\beta = 4i - 4$ reducing the count of β .

$\therefore$ Repeating the procedure we get $\beta = 0$

Hence G is cordial.

III. CONCLUSION

If ‘G’ is a 3-regular bipartite graph with partite sets ‘A’ and ‘B’ such that | A | = | B | = n
Then G is cordial for n = 4m

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